Superconformal indices for $\mathcal{N} = 1$ theories with multiple duals

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Abstract

Following a recent work of Dolan and Osborn, we consider superconformal indices of four dimensional $\mathcal{N} = 1$ supersymmetric field theories related by an electric-magnetic duality with the $SP(2N)$ gauge group and *fixed rank* flavour groups. For the $SP(2)$ (or $SU(2)$) case with 8 flavours, the electric theory has index described by an elliptic analogue of the Gauss hypergeometric function constructed earlier by the first author. Using the E_7-root system Weyl group transformations for this function, we build a number of dual magnetic theories. One of them was originally discovered by Seiberg, the second model was built by Intriligator and Pouliot, the third one was found by Csáki et al. We argue that there should be in total 72 theories dual to each other through the action of the coset group $W(E_7)/S_8$. For the general $SP(2N)$, $N > 1$, gauge group, a similar multiple duality takes place for slightly more complicated flavour symmetry groups. Superconformal indices of the corresponding theories coincide due to the Rains identity for a multidimensional elliptic hypergeometric integral associated with the BC_N-root system.

Contents

1 Introduction

One of the more important recent achievements of mathematical physics consists of the discovery of elliptic hypergeometric functions – a new class of special functions of hypergeometric type (see [1] for a survey of the corresponding results and relevant literature). These functions have found applications in the theory of Yang-Baxter equation, integrable discrete time chains, elliptic Calogero-Sutherland type models and so on. Although connection with the classical root systems has been explicitly traced in the structure of many elliptic hypergeometric functions, their group theoretical interpretation remained largely obscure.

In recent papers Römelsberger [2] and Kinney et al [3] have described topological indices for four dimensional supersymmetric conformal field theories. As suggested in [2], superconformal indices of the $\mathcal{N} = 1$ models related by Seiberg duality [4, 5] should coincide as a result of some complicated group theoretical identities. Following Römelsberger's ideas, Dolan and Osborn [6] have connected superconformal indices of a number of $\mathcal{N} = 1$ supersymmetric field theories with specific elliptic hypergeometric integrals. Corresponding dual theories have the same indices due to nontrivial identities for these integrals [1].

For example, in [7] the first author has discovered the elliptic beta integral opening the door to a new class of computable integrals. It is described by the following exact

integration formula:

$$\frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{j=1}^6 \Gamma(t_j z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{dz}{2\pi i z} = \prod_{1\le j<k\le 6} \Gamma(t_j t_k;p,q), \tag{1}$$

where six complex parameters $t_j,\ j=1,\ldots,6$, and two base variables p and q satisfy the inequalities $|p|,|q|,|t_j|<1$ and the balancing condition

$$\prod_{j=1}^6 t_j = pq.$$

Here $\mathbb{T}$ denotes the unit circle with positive orientation and

$$\Gamma(z;p,q) := \prod_{j,k=0}^\infty \frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}$$

is the elliptic gamma function. In (1) and below we denote $(t;q)_\infty := \prod_{k=0}^\infty (1-tq^k)$ and use the conventions

$$\Gamma(tz^{\pm 1};p,q) := \Gamma(tz;p,q)\Gamma(tz^{-1};p,q), \quad \Gamma(z^{\pm 2};p,q) := \Gamma(z^2;p,q)\Gamma(z^{-2};p,q),$$
$$\Gamma(tz^{\pm 1}w^{\pm 1};p,q) := \Gamma(tzw;p,q)\Gamma(tzw^{-1};p,q)\Gamma(tz^{-1}w;p,q)\Gamma(tz^{-1}w^{-1};p,q).$$

As shown by Dolan and Osborn [6], the left hand side of formula (1) describes the superconformal index of the "electric" theory with $SU(2)$ gauge group and quark superfields in the fundamental representation of the $SU(6)$ flavour group. The "magnetic" dual theory, suggested by Seiberg in [4], does not have gauge degrees of freedom; the matter sector contains meson superfields in 15-dimensional antisymmetric $SU(6)$-tensor representation of the second rank; and its superconformal index is described by the right-hand side of relation (1). This duality provides the simplest example of the so-called s-confining theories.

Seiberg duality is a fundamental concept of the modern quantum field theory [4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Corresponding models contain particular sets of fields transforming as representations of the group $G_{st} \times G \times F$, where $G_{st} = SU(2,2|1)$ is the space-time superconformal symmetry group (containing the R-symmetry subgroup $U(1)_R$ rotating supercharges), G is the local gauge invariance group, and F is the global flavour symmetry group. Conditionally, electric theories are considered as manifestations of a unique complicated "stringy" dynamics in the weak coupling regime. The magnetic theories are assigned then to the strong coupling limit. Some of the electric theories can have more than one dual magnetic partner, as was described for the first time by Intriligator and Seiberg [10] (these partners may differ by symmetries, fields content, or superpotentials).

We have considered systematically superconformal indices of known $\mathcal{N}=1$ supersymmetric theories obeying Seiberg dualities and compared them with known elliptic hypergeometric integrals. There are many dualities for G composed from $SU(N)$, $SP(2N)$, $SO(N)$, G_2 groups and F fixed as products of $SU(N_f)$ and $U(1)$ groups. For some of

them, coincidence of superconformal indices was established in [6] as a consequence of previously shown relations for integrals. As a result of our analysis, we confirm equality of such indices for several other dual theories and, additionally, we arrive at many new conjectures for different elliptic hypergeometric functions identities. Moreover, from some known integral identities, we arrive at a good number of new Seiberg dualities. In this paper we limit ourselves to the models with $G = SP(2N)$ and fixed rank flavour groups $SU(8)$ or $SU(8) \times U(1)$ and $SU(6)$ or $SU(6) \times U(1)$, and their various splits into products of $SU(4), SU(3), SU(2)$, and $U(1)$ groups. All the dualities described in this paper contain relation (1) as a special limiting case of the superconformal index equalities.

The main motivation for us to consider these particular cases of flavour groups comes from the properties of the elliptic analogue of the Gauss hypergeometric function constructed by the first author [17, 1]. This function transforms nicely under the action of the Weyl group $W(E_7)$ for the exceptional root system E_7, and it is interpreted as a superconformal index for field theories with $G = SP(2)$ (or $SU(2)$) and $F = SU(8)$. Using this fact, we conjecture existence of distinguished 72 supersymmetric field theories related to each other by the Seiberg dualities (i.e., all of them should coincide in the infrared fixed points). The first duality was discovered by Seiberg himself [5]. The second dual theory was found by Intriligator and Pouliot in [12]. The third admissible magnetic theory was discovered by Csáki et al in [15]. Here we argue for the existence of other models using different interpretation of the flavour groups. Moreover, our analysis shows that reduction of the number of flavours from 8 to 6 preserves the multiple duality phenomenon which indicates on the incompleteness of the "$N_f = N_c+1$" Seiberg duality analysis existing in the literature.

For $G = SP(2N), N > 1$, we use the generalized symmetry transformations for the type II elliptic hypergeometric integral on the BC_N-root system established by Rains [18]. These transformations are described again by the Weyl group $W(E_7)$. By interpreting the latter integral as a superconformal index, we conjecture again existence of 72 self-dual theories. Only one of the corresponding dualities was found earlier in the literature [14]. Here we present two new different classes of dualities employing the antisymmetric tensor matter field. The 't Hooft anomaly matching conditions are satisfied for all our dualities (for smaller flavour groups). The details, as well as a full list of known dual theories and related superconformal indices, are described in a separate paper [19].

2 Superconformal index

In [2] Römelsberger has constructed the superconformal index which counts BPS operators protected only by one supercharge in four dimensional $\mathcal{N} = 1$ superconformal theories. According to his analysis, first one should determine the index for single particle states which is given by the formula (for more details on the construction and the

superconformal group, see [2, 6])

$$i(t,x,h,g)=\frac{2t^2-t(x+x^{-1})}{(1-tx)(1-tx^{-1})}\chi_{adj}(g) \\ +\sum_i\frac{t^{2r_i}\chi_{R_F,i}(h)\chi_{R_G,i}(g)-t^{2-2r_i}\chi_{\bar{R}_F,i}(h)\chi_{\bar{R}_G,i}(g)}{(1-tx)(1-tx^{-1})}. \tag{2}$$

Here the first term represents contribution of gauge fields belonging to the adjoint representation of the group G. The sum $\sum_i$ runs over chiral matter fields φ_i transforming as the gauge group representations $R_{G,i}$ and flavour symmetry group representations $R_{F,i}$, with $\chi_{adj}(g)$, $\chi_{R_F,i}(h)$, and $\chi_{R_G,i}(g)$ being the appropriate characters. Logarithms of the free parameters t and x play the role of chemical potentials for particular generators of the superconformal algebra. The terms proportional to t^{2r_i} and t^{2-2r_i} result from the chiral scalar fields with the R-charges $2r_i$ and fermion descendants with $\bar{\jmath}=\frac{1}{2}$ of the conjugate anti-chiral partners whose R-charges are equal to $-2r_i$. In order to determine the index for all gauge singlet operators relevant for confining theories, formula (2) is then inserted into the "plethystic" exponential averaged over the gauge group, which yields the matrix integral

$$I(t,x,h) \;=\; \int_G d\mu(g)\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}i\big(t^n,x^n,h^n,g^n\big)\right), \tag{3}$$

where $d\mu(g)$ is the G-invariant measure. Such type of formulas appeared in computation of partition functions of different statistical mechanics models and quantum field theories, see, e.g., [20, 3, 21] and [22] (where this algorithm was referred to as the "plethystic program") or [23].

Suppose that we have a chiral superfield with some $U(1)$ symmetry. Then the corresponding parameter r in the above formula is replaced by $r+s$, where s is an arbitrary chemical potential associated with the generator of $U(1)$. It is convenient to introduce new variables

$$p=tx,\quad q=tx^{-1},\quad z=t^{2s},\quad y=t^{2r}z,$$

and to assume that p,q are real and $0\le q,p<1$. Then the single particle states index takes the form

$$i_S(p,q,y)=\frac{t^{2r}z-t^{2-2r}z^{-1}}{(1-tx)(1-tx^{-1})}=\frac{y-pq/y}{(1-p)(1-q)}. \tag{4}$$

As a result of the described index building algorithm, one obtains the elliptic gamma function [2]

$$\Gamma(y;p,q) \;=\; \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}i_S(p^n,q^n,y^n)\right)=\prod_{j,k=0}^{\infty}\frac{1-y^{-1}p^{j+1}q^{k+1}}{1-y\,p^j\,q^k}. \tag{5}$$

This is precisely how $\Gamma(y;p,q)$ emerged in the partition function asymptotics for Baxter's eight vertex model [1]. For the gauge field part one can set

$$i_V(p,q) \;=\; \frac{2t^2-t(x+x^{-1})}{(1-tx)(1-tx^{-1})}=-\frac{p}{1-p}-\frac{q}{1-q}=1-\frac{1-pq}{(1-p)(1-q)}.$$

Since for $SP(2)$ (or $SU(2)$) gauge group one has $\chi_{adj}(g) = z^2 + z^{-2} + 1$, the algorithm yields for different pieces of this character

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_V(p^n, q^n)(z^{2n} + z^{-2n})\right) = \frac{\theta(z^2;p)\theta(z^2;q)}{(1-z^2)^2}$$
$$= \frac{1}{(1-z^2)(1-z^{-2})\Gamma(z^{\pm 2};p,q)},$$
$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i_V(p^n, q^n)\right) = (p;p)_\infty (q;q)_\infty,$$

where the theta function is defined as

$$\theta(z;p) \ = (z;p)_\infty (pz^{-1};p)_\infty = \prod_{j=0}^{\infty} (1 - zp^j)(1 - z^{-1}p^{j+1}). \tag{6}$$

3 Multiple duality for $SP(2)$ gauge group

3.1 Electric theory with the flavour group $F = SU(8)$

In this section we consider multiple duality phenomenon for a particular electric theory defined as supersymmetric QCD with the internal symmetry group $G \times F$, where

$$G \ = \ SP(2), \qquad F \ = SU(8).$$

All $\mathcal{N} = 1$ supersymmetric theories have the global R-symmetry described by $U(1)_R$-group. So, in the taken version of SQCD, we have one chiral scalar multiplet Q belonging to the fundamental representations (denoted as f) of $SP(2)$ and $SU(8)$, and the vector multiplet V in the adjoint representation (denoted as adj) of $SP(2)$ without coupling to $SU(8)$. We gather information about properties of the fields in Table 1, where we provide values of r_i for the $U(1)_R$-group in the last column.

Table 1.

	$SP(2)$	$SU(8)$	$U(1)_R$
Q	f	f	$\frac{1}{4}$
V	adj	1	$\frac{1}{2}$

Characters $\chi_R(g)$ for $g \in SP(2)$ are functions of one complex variable z, while the characters $\chi_R(h)$ for $h \in SU(8)$ are functions of eight complex variables

$$y = (y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8), \qquad \prod_{i=1}^{8} y_i \ = \ 1.$$

The single particle state index is given by the expression

$$i_E(p,q,z,y) = -\left(\frac{p}{1-p} + \frac{q}{1-q}\right)\chi_{SP(2),adj}(z) \tag{7}$$
$$+ \frac{1}{(1-p)(1-q)}\left((pq)^r \chi_{SU(8),f}(y)\chi_{SP(2),f}(z) - (pq)^{1-r}\chi_{SU(8),\overline{f}}(y)\chi_{SP(2),\overline{f}}(z)\right),$$

where $2r = 1/2$ is the $R-$charge of the scalar component of the field Q. The electric index is given then by the following integral (corresponding characters can be found in the Appendix, and we borrow the matrix group measures from [6]):

$$I_E = \frac{(p;p)_\infty (q;q)_\infty}{2} \int_{\mathbb{T}} \frac{\prod_{i=1}^8 \Gamma((pq)^{1/4} y_i z^{\pm 1}; p,q)}{\Gamma(z^{\pm 2}; p,q)} \frac{dz}{2\pi i z}. \tag{8}$$

In [17] the first author has constructed the following elliptic hypergeometric function

$$I(t_1,\ldots,t_8;p,q) = \kappa \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(t_j z^{\pm 1}; p,q)}{\Gamma(z^{\pm 2}; p,q)} \frac{dz}{z}, \qquad \kappa = \frac{(p;p)_\infty (q;q)_\infty}{4\pi i}, \tag{9}$$

with the constraints $|t_j| < 1$ for eight complex variables $t_1,\ldots,t_8 \in \mathbb{C}$ and the balancing condition $\prod_{j=1}^8 t_j = (pq)^2$. This integral is interpreted as a natural elliptic analogue of the Gauss hypergeometric function since it has many classical properties [1]. In particular, it obeys the following symmetry transformation derived in [17] (see there formula (6.11) for $n = 1$)

$$I(t_1,\ldots,t_8;p,q) = \prod_{1\le j<k\le 4} \Gamma(t_j t_k; p,q) \Gamma(t_{j+4} t_{k+4}; p,q)\, I(s_1,\ldots,s_8;p,q), \tag{10}$$

where complex variables s_j, $|s_j| < 1$, are connected with t_j, $j = 1,\ldots,8$, as follows

$$\begin{aligned} s_j &= \rho^{-1} t_j,\ j = 1,2,3,4, \quad s_j = \rho t_j,\ j = 5,6,7,8, \\ \rho &= \sqrt{\frac{t_1 t_2 t_3 t_4}{pq}} = \sqrt{\frac{pq}{t_5 t_6 t_7 t_8}}. \end{aligned} \tag{11}$$

This fundamental relation extends the evident S_8-permutational group of symmetries of the integral in parameters t_j to the Weyl group $W(E_7)$ of the exceptional root system E_7 [18].

Evidently, integral (9) coincides with the electric superconformal index after appropriate change of variables. In the following sections we use formula (10) as a base for establishing equalities of superconformal indices in known simplest Seiberg dual theories, as well as for the discovery of new dualities.

Let e_i, $i = 1,\ldots,8$, form an orthonormal basis of the Euclidean space $\mathbb{R}^8$. Denoting as $\langle x,y \rangle$ the scalar product in this space, we have $\langle e_i, e_j \rangle = \delta_{ij}$. The root system A_7 consists of the vectors $v = \{e_i - e_j,\ i \neq j\}$, and its Weyl group S_8 (of dimension 8!) is generated by the reflections

$$x \to R_v(x) = x - \frac{2\langle v,x \rangle}{\langle v,v \rangle} v, \tag{12}$$

acting in the hyperplane orthogonal to the vector $\sum_{i=1}^8 e_i$. This hyperplane vectors $x = \sum_{i=1}^8 x_i e_i \in \mathbb{R}^8$ satisfy the constraint $\sum_{i=1}^8 x_i = 0$. Evidently, $R_v(\lambda v) = -\lambda v$ for any $\lambda \in \mathbb{C}$ and $R_v^2 = 1$.

Consider now the change of variables $t_j = e^{2\pi i x_j}(pq)^{1/4}$ in integral (9), which automatically satisfies the balancing condition. The transformation of parameters in (10) corresponds then to the reflection $R_v(x)$ with respect to the vector $v = \frac{1}{2}(\sum_{i=1}^4 e_i - \sum_{i=5}^8 e_i)$ of the length $\langle v, v\rangle = 2$ belonging to the root system E_7:

$$x' = (x'_1, \ldots, x'_8) = (x_1 - \delta, \ldots, x_4 - \delta, x_5 + \delta, \ldots, x_8 + \delta), \quad \delta = \frac{1}{2}\sum_{i=1}^4 x_i. \tag{13}$$

This is the key reflection generating together with S_8 the group $W(E_7)$.

Let us apply now S_8-group to the set $\{x'\} = R_v(S_8(x))$. Clearly, the action of its $S_4 \times S_4$-subgroup leads to the vectors that can be obtained by permutation of $x_1, \ldots, x_8$ in (13). However, if we mix coordinates of x' from $\sigma_1 := \{x'_1, \ldots, x'_4\}$ and $\sigma_2 := \{x'_5, \ldots, x'_8\}$, we arrive at new vectors x''. $16 \times 8!$ of them are obtained by permutation by one coordinate from σ_1 and σ_2. $18 \times 8!$ new vectors appear from permutation by two coordinates from σ_1 and σ_2 (modulo permutation of σ_1 and σ_2 themselves which does not lead to new vectors).

Applying again to the derived set of vectors the key reflection with respect to v, we find a number of new elements of the $W(E_7)$-group orbit. For instance, we obtain

$$\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_8) = (-x_1 + \delta, \ldots, -x_4 + \delta, -x_5 - \delta, \ldots, -x_8 - \delta). \tag{14}$$

Application of the S_8-group to these new elements yields another set of $(16 + 18) \times 8!$ new vectors. Finally, a third application of the key R_v-reflection yields one more set of independent $8!$ vectors obtained by coordinate permutations of $\tilde{x}' = (-x_1, \ldots, -x_8)$. This consideration shows that the dimension of $W(E_7)$ is $72 \times 8!$ with the coset $W(E_7)/S_8$ consisting of 72 elements generating transformations $x \to x' \to x'' \to \tilde{x} \to \ldots$ of the described above form (including the identity transformation).

3.2 First class of dualities with $F = SU(4) \times SU(4) \times U(1)_B$

Using relation (10), we obtain the first magnetic theory with the internal symmetry groups

$$G = SP(2), \qquad F = SU(4)_l \times SU(4)_r \times U(1)_B. \tag{15}$$

It has two chiral scalar multiplets q and $\widetilde{q}$ belonging to the fundamental representation of $SP(2)$-group, the gauge field in the adjoint representation $\widetilde{V}$, and the singlets M and $\widetilde{M}$ in the antisymmetric tensor representations of $SU(4)$-group. Properties of the fields are summarized in Table 2.

Table 2.

	$SP(2)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	f	1	-1	$\frac{1}{4}$
$\widetilde{q}$	f	1	f	1	$\frac{1}{4}$
M	1	T_A	1	2	$\frac{1}{2}$
$\widetilde{M}$	1	1	T_A	-2	$\frac{1}{2}$
$\widetilde{V}$	adj	1	1	0	$\frac{1}{2}$

This theory was found by Csáki et al in [15], where it was listed as the third dual theory for the $SU(2)$ gauge group. It differs from the original $SU(2)$ duality found by Seiberg [5], to be described below.

The single particle states index for this magnetic dual theory is given by the expression

$$
\begin{aligned}
i_M(p,q,z,\widetilde{y},\widehat{y}) = & -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{SP(2),adj}(z) \\
& + \frac{1}{(1-p)(1-q)}\Big((pq)^{r_q}\frac{1}{v}\chi_{SU(4),f}(\widetilde{y})\chi_{SP(2),f}(z) - (pq)^{1-r_q}v\chi_{SU(4),\overline{f}}(\widetilde{y})\chi_{SP(2),\overline{f}}(z) \\
& + (pq)^{r_M}v^2\chi_{SU(4),T_A}(\widetilde{y}) - (pq)^{1-r_M}\frac{1}{v^2}\chi_{SU(4),\overline{T}_A}(\widetilde{y}) \\
& + (pq)^{r_{\widetilde{q}}}v\chi_{SU(4),f}(\widehat{y})\chi_{SP(2),f}(z) - (pq)^{1-r_{\widetilde{q}}}\frac{1}{v}\chi_{SU(4),\overline{f}}(\widehat{y})\chi_{SP(2),\overline{f}}(z) \\
& + (pq)^{r_{\widetilde{M}}}\frac{1}{v^2}\chi_{SU(4),T_A}(\widehat{y}) - (pq)^{1-r_{\widetilde{M}}}v^2\chi_{SU(4),\overline{T}_A}(\widehat{y})\Big),
\end{aligned}
\tag{16}
$$

where the values of all r's can be read off from the last column of Table 2. Arbitrary variable v is associated with the $U(1)_B$-group, its powers are determined by the baryonic charges of the fields. The characteristic variables $\widetilde{y}_j$ and $\widehat{y}_j$ of the $SU(4)$-groups satisfy the constraints $\prod_{j=1}^4 \widetilde{y}_j = \prod_{j=1}^4 \widehat{y}_j = 1$.

In order to compare superconformal indices of the electric and magnetic theories we need matching of the characteristic variables of two different flavour groups. We denote

$$
\widetilde{y}_j \ = \ v^{-1}y_j, \qquad \widehat{y}_j \ = \ vy_{j+4}, \quad j=1,2,3,4,
$$

and set

$$
v = \sqrt[4]{y_1y_2y_3y_4}, \qquad v^{-1} = \sqrt[4]{y_5y_6y_7y_8}.
$$

Applying now formula (3), we obtain the superconformal index for the magnetic theory

$$
\begin{aligned}
I_M^{(1)} \ = \ & \frac{(p;p)_\infty(q;q)_\infty}{2}\prod_{1\le i<j\le 4}\Gamma((pq)^{r_M}y_iy_j;p,q)\prod_{5\le i<j\le 8}\Gamma((pq)^{r_{\widetilde{M}}}y_iy_j;p,q) \\
& \times \int_{\mathbb{T}} \frac{\prod_{i=1}^4\Gamma((pq)^{r_q}v^{-2}y_iz^{\pm1};p,q)\prod_{i=5}^8\Gamma((pq)^{r_{\widetilde{q}}}v^2y_iz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{2\pi iz}.
\end{aligned}
\tag{17}
$$

Using the key formula (10), we find $I_E = I_M^{(1)}$. This is a new confirmation of the equality of superconformal indices for Seiberg dual theories, additional to the results of [6].

3.3 Second class of dualities with $F = SU(4)\times SU(4)\times U(1)_B$

This dual model has the same flavour group as in the previous section and two chiral scalar multiplets q and $\widetilde{q}$ belonging to the fundamental representation of $SP(2)$, gauge field in the adjoint representation $\widetilde{V}$, and a singlet M. This is the original Seiberg duality for $SU(2)$ group [5] (it corresponds also to the first $SU(2)$ dual model in [15]). The representation content of the model is summarized in Table 3.

Table 3.

	$SP(2)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)_R$
q	f	$\overline{f}$	1	1	$\frac{1}{4}$
$\widetilde{q}$	f	1	$\overline{f}$	-1	$\frac{1}{4}$
M	1	f	f	0	$\frac{1}{2}$
$\widetilde{V}$	adj	1	1	0	$\frac{1}{2}$

The characteristic variables for the $SU(4)$ subgroups are chosen in the same way as in the previous case.

The single particle index is

$$
\begin{aligned}
i_M(p,q,z,\widetilde{y},\widehat{y}) = & -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{SP(2),adj}(z) \\
& + \frac{1}{(1-p)(1-q)}\Big((pq)^{r_q} v\chi_{SU(4),f}(\widetilde{y})\chi_{SP(2),f}(z) - (pq)^{1-r_q}\frac{1}{v}\chi_{SU(4),\overline{f}}(\widetilde{y})\chi_{SP(2),\overline{f}}(z) \\
& + (pq)^{r_{\widetilde{q}}}\frac{1}{v}\chi_{SU(4),f}(\widehat{y})\chi_{SP(2),f}(z) - (pq)^{1-r_{\widetilde{q}}} v\chi_{SU(4),\overline{f}}(\widehat{y})\chi_{SP(2),\overline{f}}(z) \\
& + (pq)^{r_M}\chi_{SU(4),f}(\widetilde{y})\chi_{SU(4),f}(\widehat{y}) - (pq)^{1-r_M}\chi_{SU(4),\overline{f}}(\widetilde{y})\chi_{SU(4),\overline{f}}(\widehat{y})\Big).
\end{aligned} \tag{18}
$$

The superconformal index itself in this magnetic theory is found to be

$$
\begin{aligned}
I_M^{(2)} = & \frac{(p;p)_\infty (q;q)_\infty}{2}\prod_{i=1}^{4}\prod_{j=5}^{8}\Gamma((pq)^{r_M}y_iy_j;p,q) \\
& \times \int_{\mathbb{T}} \frac{\prod_{i=1}^4 \Gamma((pq)^{r_q}v^2y_i^{-1}z^{\pm 1};p,q)\prod_{i=5}^8\Gamma((pq)^{r_{\widetilde{q}}}v^{-2}y_i^{-1}z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)}\frac{dz}{2\pi i z}.
\end{aligned} \tag{19}
$$

Equality $I_E = I_M^{(2)}$ is a direct consequence of transformation (10). Namely, it is necessary to repeat once more this transformation with the parameters s_3, s_4, s_5, s_6 playing the role of t_1, t_2, t_3, t_4 and permute appropriately parameters in the result (see, e.g., [1]). Note that this match of superconformal indices was obtained also in [6] as the $N = \tilde{N} = 2$ subcase of the $SU(N) \leftrightarrow SU(\tilde{N})$ gauge group duality (see equality (6.12) there).

3.4 Third dual picture. Flavor group $SU(8)$

The third type of dual magnetic theories consists of only one model which was considered by Intriligator and Pouliot [12] (it was described as the second $SU(2)$ dual in [15]). It has the following symmetry groups

$$G = SP(2), \qquad F = SU(8)$$

differing from the previous cases. There is one chiral scalar multiplet q in the fundamental representation of $SP(2)$ and antifundamental representation $\bar{f}$ of $SU(8)$, the gauge field in the adjoint representation $\widetilde{V}$, and one singlet M, as described in Table 4.

Table 4.

	$SP(2)$	$SU(8)$	$U(1)_R$
q	f	$\overline{f}$	$\frac{1}{4}$
M	1	T_A	$\frac{1}{2}$
$\widetilde{V}$	adj	1	$\frac{1}{2}$

The single state index in this case is

$$i_M(p,q,z,y) = -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{SP(2N),adj}(z) \tag{20}$$
$$+\frac{1}{(1-p)(1-q)}\Bigg\{(pq)^{\tilde{r}}\chi_{SU(8),\overline{f}}(y)\chi_{SP(2N),f}(z)-(pq)^{1-\tilde{r}}\chi_{SU(8),f}(y)\chi_{SP(2N),\overline{f}}(z)$$
$$+(pq)^{r_M}\chi_{SU(8),T_A}(y)-(pq)^{1-r_M}\chi_{SU(8),\overline{T}_A}(y)\Bigg\},$$

The magnetic index is easily computed to be given by the integral

$$I_M^{(3)} = \frac{(p;p)_\infty(q;q)_\infty}{2}\prod_{1\le i<j\le 8}\Gamma((pq)^{r_M}y_iy_j;p,q)\int_{\mathbb{T}}\frac{\prod_{i=1}^{8}\Gamma((pq)^{\tilde{r}}y_i^{-1}z^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{2\pi iz}. \tag{21}$$

The equality $I_E = I_M^{(3)}$ follows from the already established relation $I_M^{(1)} = I_M^{(2)}$, which is, in a sense, a third sequential application of transformation (10) intertwined with the S_8-group actions (see, e.g., [1]). The derived match of superconformal indices coincides also with the consideration of $N=\tilde{N}=1$ case of the $SP(2N)\leftrightarrow SP(2\tilde{N})$ gauge group duality in [6] (see equality (7.12) there).

3.5 Discussion of the number of dualities and some puzzles

We have seen that there are at least four field theories dual to each other, and whose superconformal indices are connected by the specific Weyl group transformations for the exceptional root system E_7. Such transformations are determined by the coset $W(E_7)/S_8$ of dimension 72. Logically one would expect therefore bigger number of dualities than we have exhibited.

Trying to model these additional dualities, we considered the flavour symmetry group

$$F = SU(3)_l\times U(1)_1\times SU(3)_r\times U(1)_2\times U(1)_B \tag{22}$$

and the gauge theory with the field content fixed in Table 5.

	$SP(2)$	$SU(3)$	$U(1)_1$	$SU(3)$	$U(1)_2$	$U(1)_B$	$U(1)_R$
q_1	f	1	$\frac{3}{2}$	1	$\frac{3}{2}$	2	$\frac{1}{4}$
q_2	f	f	$\frac{1}{2}$	1	$-\frac{3}{2}$	0	$\frac{1}{4}$
q_3	f	1	$\frac{3}{2}$	1	$\frac{3}{2}$	-2	$\frac{1}{4}$
q_4	f	1	$-\frac{3}{2}$	f	$\frac{1}{2}$	0	$\frac{1}{4}$
X_1	1	1	0	$\overline{f}$	-2	-2	$\frac{1}{2}$
X_2	1	$\overline{f}$	-2	1	0	2	$\frac{1}{2}$
M_2	1	$\overline{T}_A$	-1	1	3	0	$\frac{1}{2}$
M_4	1	1	3	$\overline{T}_A$	-1	0	$\frac{1}{2}$
$\tilde{V}$	adj	1	0	1	0	0	$\frac{1}{2}$

Table 5. Additional dualities of the first class.

It is possible to build the superconformal index for this model and find that it matches with the second class index (17). However, as it was pointed to us by A. Khmelnitsky, here one actually has a theory with the flavour group $F' = SU(4)'_l \times SU(4)'_r \times U(1)'_B$. Let us take the dual theory of Sect. 3.2 with the flavour group F' and consider decomposition of the corresponding fields with respect to the subgroup $SU(3)_l \times U(1)'_1 \times SU(3)_r \times U(1)'_2 \times U(1)'_B \subset F'$ (evidently, there are more than one such subgroup). Using the fact that for $SU(3)$ group the T_A and $\bar{f}$ representations are Hodge equivalent, one obtains the theory described in Table 5, provided hypercharges of the corresponding $U(1)$ groups are identified as follows:

$$Q'_B = \frac{1}{2}(Q_B + Q_2 - Q_1),$$
$$Q'_1 = -\frac{1}{12}Q_1 + \frac{1}{4}(Q_B - Q_2), \quad Q'_2 = -\frac{1}{12}Q_2 - \frac{1}{4}(Q_B + Q_1).$$

Similarly one can consider a dual theory with the field content fixed in Table 6, belonging to the second class of dualities since its superconformal index matches with (19).

	$SP(2)$	$SU(3)$	$U(1)_1$	$SU(3)$	$U(1)_2$	$U(1)_B$	$U(1)_R$
q_1	f	1	$-\frac{3}{2}$	1	$-\frac{3}{2}$	-2	$\frac{1}{4}$
q_2	f	$\overline{f}$	$-\frac{1}{2}$	1	$\frac{3}{2}$	0	$\frac{1}{4}$
q_3	f	1	$-\frac{3}{2}$	1	$-\frac{3}{2}$	2	$\frac{1}{4}$
q_4	f	1	$\frac{3}{2}$	$\overline{f}$	$-\frac{1}{2}$	0	$\frac{1}{4}$
X_1	1	f	-1	f	-1	0	$\frac{1}{2}$
X_2	1	1	3	1	3	0	$\frac{1}{2}$
Y_1	1	f	2	1	0	2	$\frac{1}{2}$
Y_2	1	1	0	f	2	-2	$\frac{1}{2}$
$\tilde{V}$	adj	1	0	1	0	0	$\frac{1}{2}$

Table 6. Additional dualities of the second class.

Again, one can embed this model into the theory with F' flavour group with the same relation between $U(1)$-charges as above. One could claim that the theories of Tables 5 and 6 do not differ from models of Sects. 3.2 and 3.3, respectively. However, they differ by the anomaly matching conditions. Here it is necessary first to explain how we compare global anomalies of dual theories. Electric theory and third class dual models have the same $SU(8)$ flavour group and there are no problems in comparison of anomalies. However in the models of first and second classes the flavour groups are $SU(4)_l \times SU(4)_r \times U(1)_B$ which leads to the main puzzle of these dualities. According to 't Hooft, anomalies of the global symmetries should match in the ultraviolet (UV) and infrared (IR) regimes. In the second and third class dual models, which supposedly describe the same IR dynamics, we miss a large piece of the $SU(8)$ axial currents needed for comparison with the UV picture of the electric theory. Surprisingly, this problem was not discussed in the literature although in many papers this mismatch in flavour groups for $SU(2)$ gauge group models was noticed (including the original Seiberg work [5]). We have found only one paper by Leigh and Strassler [24] with partial discussion of the dynamics in the presence of such an "accidental symmetry".

So, in [24] it is claimed that at the IR fixed point the original Seiberg dual model has actually full $SU(8)$ flavour group, a part of which is realized in some non-linear non-perturbative way. In support of this conjecture, rotations of a pair of quark superfields with mass terms added to the electric theory was considered. In the dual picture a superpotential was suggested depending on the parameters of this rotation, indicating on the existence of continuously many dual theories. However, one bothering issue with considerations of [24] is that the manifest flavour symmetry group is changing its structure abruptly with vanishing of one of the superpotential parameters. Second, more important, no explicit flavour $SU(8)$-transformations of the dual theory were exhibited, their influence on the whole superpotential (e.g., without adding mass terms) was not established, and no 't Hooft anomaly matching conditions were verified for the missing part of the global symmetry currents. All these puzzles show that understanding of the duality for the $SU(2)$ gauge group, where one has an "accidentally" large flavour group, is not satisfactory yet.

Return now to the model of Sect. 3.2 with the flavour group F. It is not difficult to check [15] that its anomalies match with the anomalies of electric theory for the subgroup $F \subset SU(8)$. Similar picture holds evidently for the model of Table 5 since it is equivalent to a similar model with the flavour group F'. However, if we compare anomalies of the Csáki et al and Table 5 models, there is a nontrivial possibility to identify the $U(1)_B$ group in Table 5 (which differs from $U(1)'_B$) with the $U(1)_B$ in Table 2. To compare anomalies of these two dual models, we need to decompose fields in Table 2 with respect to the flavour group of Table 5. After that it can be checked that the anomalies do match indeed. It looks like that these two Csáki et al type models are related to each other by some $SU(8)$ flavour space rotation supporting again the Leigh-Strassler claim about the presence of this hidden symmetry at the IR fixed point. However, we cannot describe the explicit form of this rotation. Similar picture holds for the Seiberg type second class dual models of Tables 3 and 6.

Moreover, one can consider other subgroups of the group $SU(4)_l \times SU(4)_r \times U(1)_B$:

$$
\begin{gathered}
(SU(2) \times SU(2) \times U(1))^2 \times U(1)_B, \quad U(1)^3 \times SU(2) \times SU(3) \times U(1)_B, \\
U(1)^2 \times SU(2)^3 \times U(1)_B, \quad U(1)^4 \times SU(2)^2 \times U(1)_B, \\
U(1)^5 \times SU(2) \times U(1)_B, \quad U(1)^6 \times U(1)_B
\end{gathered}
$$

and verify anomaly matchings for them. Relying on the structure of the coset space $W(E_7)/S_8$ described in the end of Sect. 3.1, we expect that there will be 35 theories in both first and second classes of dualities. A diagonal $SU(8)$ matrix can be split into two 4×4 matrices with different entries (up to permutation of these submatrices) in $\frac{1}{2}\binom{8}{4} = 35$ ways. This qualitative counting corresponds to the number of ways one can embed $SU(4)_l \times SU(4)_r \times U(1)_B$ into the $SU(8)$ group.

Therefore we expect that the total number of theories distinguished in UV and related by the duality is equal to 72. In order to clarify the situation completely, one has to build superpotentials differentiating all these models. Also, one may try to build non-linear chiral models for degrees of freedom associated with the cosets $SU(8)/(SU(4) \times SU(4) \times U(1))$ such that the full anomaly matching conditions will be restored pairwise for all 72 models. Discussion of such questions lies beyond the scope of the present paper.

4 Multiple duality for higher rank symplectic gauge groups

4.1 Electric theory with the flavour group $SU(8)\times U(1)$

Now we pass to investigation of the general $SP(2N)$ gauge group models. We describe the same multiple duality phenomenon for $\mathcal{N}=1$ SQCD electric theory with the overall internal symmetry group $G\times F$, where

$$G \ = \ SP(2N), \quad N>1, \qquad F \ = SU(8)\times U(1).$$

This theory has one chiral scalar multiplet Q belonging to the fundamental representations of G and F, the vector multiplet V in the adjoint representation, and the anti-symmetric $SP(2N)$-tensor field X. The field content is fixed in Table 7.

Table 7.

	$SP(2N)$	$SU(8)$	$U(1)$	$U(1)_R$
Q	f	f	$-\frac{N-1}{4}$	$\frac{1}{4}$
X	T_A	1	1	0
V	adj	1	0	$\frac{1}{2}$

For $N=1$ the field X is absent and $U(1)$-group is completely decoupled.

This electric theory and its one magnetic dual were considered in [14]. However, there are more dualities similar to the $SP(2)$ group case. The single particle states index is

$$\begin{aligned} i_E(p,q,z,y) = &-\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{SP(2N),adj}(z) \\ &+\frac{1}{(1-p)(1-q)}\Big\{(pq)^{r_X}\chi_{SP(2N),T_A}(z)-(pq)^{1-r_X}\chi_{SP(2N),\overline{T}_A}(z) \\ &+(pq)^{r_Q}\chi_{SU(8),f}(y)\chi_{SP(2N),f}(z)-(pq)^{1-r_Q}\chi_{SU(8),\overline{f}}(y)\chi_{SP(2N),\overline{f}}(z)\Big\}, \end{aligned} \tag{23}$$

where characters $\chi_R(g)$ for $g\in SP(2N)$ are functions of free N complex variables $z_j,\ j=1,\ldots,N$. We denote also

$$r_Q \ = \ R_Q+e_Qs, \qquad r_X \ = \ e_Xs,$$

where $2R_Q=1/2$ is the R-charge of the Q-field, $e_Q=-(N-1)/4$ and $e_X=1$ are the $U(1)$-group hypercharges, and s is an arbitrary chemical potential for the latter abelian group. The electric superconformal index is then

$$\begin{aligned} I_E = &\frac{(p;p)_\infty^N(q;q)_\infty^N}{2^NN!}\Gamma((pq)^s;p,q)^{N-1}\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{\Gamma((pq)^sz_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)} \\ &\times\prod_{j=1}^N\frac{\prod_{k=1}^8\Gamma((pq)^{r_Q}y_kz_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{2\pi i z_j}. \end{aligned} \tag{24}$$

We have the constraint $\prod_{k=1}^8 y_k = 1$, which coincides with the balancing condition for this elliptic hypergeometric integral due to a special choice of the R-charge of the chiral scalar multiplet Q.

Now we construct a number of $SP(2N)$-dual theories from the Rains symmetry transformation [18] for the following higher rank BC_N-root system generalization of integral (9):

$$I(t_1,\ldots,t_8;t,p,q) = \prod_{1\le j<k\le 8}\Gamma(t_jt_k;p,q,t)\frac{(p;p)_\infty^N(q;q)_\infty^N}{2^NN!}$$
$$\times\int_{\mathbb{T}^N}\prod_{1\le j<k\le N}\frac{\Gamma(tz_j^{\pm1}z_k^{\pm1};p,q)}{\Gamma(z_j^{\pm1}z_k^{\pm1};p,q)}\prod_{j=1}^N\frac{\prod_{k=1}^8\Gamma(t_kz_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2})}\frac{dz_j}{2\pi iz_j}, \tag{25}$$

where nine variables $t,t_1,\ldots,t_8\in\mathbb{C}$ satisfy the balancing condition

$$t^{2N-2}\prod_{j=1}^8 t_j \;=\; (pq)^2$$

and the inequalities $|t|,|t_j|<1$. Here

$$\Gamma(z;p,q,t) \;=\; \prod_{j,k,l=0}^\infty(1-zt^jp^kq^l)(1-z^{-1}t^{j+1}p^{k+1}q^{l+1})$$

is the elliptic gamma function of the second order satisfying the key t-difference equation

$$\Gamma(tz;p,q,t) \;=\; \Gamma(z;p,q)\Gamma(z;p,q,t).$$

Rains has proved the following $W(E_7)$-group transformation for integrals (25):

$$I(t_1,\ldots,t_8;t,p,q) \;=\; I(s_1,\ldots,s_8;t,p,q), \tag{26}$$

where we denoted the variables

$$s_j=\rho^{-1}t_j,\ j=1,2,3,4,\qquad s_j=\rho t_j,\ j=5,6,7,8, \tag{27}$$
$$\rho=\sqrt{\frac{t_1t_2t_3t_4}{pqt^{1-N}}}=\sqrt{\frac{pqt^{1-N}}{t_5t_6t_7t_8}}.$$

We describe a group theoretical interpretation of integral (25) and use relation (26) for equating superconformal indices of the dual theories. We conjecture again that there are 72 theories dual to each other with only 4 of them looking essentially different.

4.2 First class of dualities

The first magnetic theory has the symmetry groups

$$G \;=\; SP(2N),\qquad F \;=\; SU(4)_l\times SU(4)_r\times U(1)_B\times U(1).$$

It contains two chiral scalar multiplets q and $\widetilde{q}$ belonging to the fundamental representations of $SP(2N)$, gauge field in the adjoint representation $\widetilde{V}$, the anti-symmetric tensor representation $\widetilde{Y}$, and the singlets M_J and $\widetilde{M}_J$, $J=0,\ldots,N-1$, as described in Table 8. Similar to $N=1$ case, we expect that there are 35 dual models in this class.

Table 8.

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)$	$U(1)_R$
q	f	f	1	-1	$-\frac{N-1}{4}$	$\frac{1}{4}$
$\widetilde{q}$	f	1	f	1	$-\frac{N-1}{4}$	$\frac{1}{4}$
Y	T_A	1	1	0	1	0
M_J	1	T_A	1	2	$\frac{2J-N+1}{2}$	$\frac{1}{2}$
$\widetilde{M}_J$	1	1	T_A	-2	$\frac{2J-N+1}{2}$	$\frac{1}{2}$
$\widetilde{V}$	adj	1	1	0	0	$\frac{1}{2}$

In this and all other tables given below the capital index J takes the values $0,\ldots,N-1$, which is not mentioned further for saving space.

The single particle state index is

$$
\begin{aligned}
i_M(p,q,z,\widetilde{y},\widehat{y}) = &-\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{SP(2N),adj}(z) \\
&+\frac{1}{(1-p)(1-q)}\Bigg\{(pq)^{r_Y}\chi_{SP(2N),T_A}(z)-(pq)^{1-r_Y}\chi_{SP(2N),\overline{T}_A}(z) \\
&+(pq)^{r_q}\frac{1}{v}\chi_{SU(4),f}(\widetilde{y})\chi_{SP(2N),f}(z)-(pq)^{1-r_q}v\chi_{SU(4),\overline{f}}(\widetilde{y})\chi_{SP(2N),\overline{f}}(z) \\
&+(pq)^{r_{\widetilde{q}}}v\chi_{SU(4),f}(\widehat{y})\chi_{SP(2N),f}(z)-(pq)^{1-r_{\widetilde{q}}}\frac{1}{v}\chi_{SU(4),\overline{f}}(\widehat{y})\chi_{SP(2N),\overline{f}}(z) \\
&+\sum_{J=0}^{N-1}\Bigg((pq)^{r_{M_J}}v^2\chi_{SU(4),T_A}(\widetilde{y})-(pq)^{1-r_{M_J}}\frac{1}{v^2}\chi_{SU(4),\overline{T}_A}(\widetilde{y}) \\
&\quad+(pq)^{r_{\widetilde{M}_J}}\frac{1}{v^2}\chi_{SU(4),T_A}(\widehat{y})-(pq)^{1-r_{\widetilde{M}_J}}v^2\chi_{SU(4),\overline{T}_A}(\widehat{y})\Bigg)\Bigg\},
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
r_q &= R_q-\frac{N-1}{4}s, \quad r_{\widetilde{q}} = R_{\widetilde{q}}-\frac{N-1}{4}s, \quad r_Y = s, \\
r_{M_J} &= R_{M_J}-\frac{1}{2}(N-1-2J)s, \quad r_{\widetilde{M}_J} = R_{\widetilde{M}_J}-\frac{1}{2}(N-1-2J)s.
\end{aligned}
$$

For the comparison with the electric theory we denote the characteristic variables as $v=\sqrt[4]{y_1y_2y_3y_4}$ and $\widetilde{y}_j = v^{-1}y_j$, $\widehat{y}_j = vy_{j+4}$, $j=1,2,3,4$. As a result, we find the superconformal index in this magnetic theory

$$
\begin{aligned}
I_M^{(1)} = &\prod_{J=0}^{N-1}\prod_{1\le i<j\le 4}\Gamma((pq)^{r_{M_J}}y_iy_j;p,q)\prod_{5\le i<j\le 8}\Gamma((pq)^{r_{\widetilde{M}_J}}y_iy_j;p,q) \\
&\times\Gamma((pq)^s;p,q)^{N-1}\frac{(p;p)_\infty^N(q;q)_\infty^N}{2^NN!}\int_{\mathbb{T}^N}\prod_{1\le i<j\le N}\frac{\Gamma((pq)^sz_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)} \\
&\times\prod_{j=1}^N\frac{\prod_{i=1}^4\Gamma((pq)^{r_q}v^{-2}y_iz_j^{\pm1};p,q)\prod_{i=5}^8\Gamma((pq)^{r_{\widetilde{q}}}v^2y_iz_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{2\pi iz_j}.
\end{aligned} \tag{29}
$$

The equality $I_E \;=\; I_M^{(1)}$ follows from the Rains transformation (26) after using the relations

$$\prod_{1\leq j<k\leq 4}\Gamma\left(\rho^{-2}t_jt_k;p,q,t\right)=\prod_{1\leq j<k\leq 4}\Gamma\left(\frac{pqt^{1-N}}{t_1t_2t_3t_4}t_jt_k;p,q,t\right)$$
$$=\prod_{1\leq j<k\leq 4}\Gamma\left(\frac{pqt^{1-N}}{t_jt_k};p,q,t\right)=\prod_{1\leq j<k\leq 4}\Gamma\left(t^Nt_jt_k;p,q,t\right)$$
$$=\prod_{1\leq j<k\leq 4}\left(\prod_{l=0}^{N-1}\Gamma\left(t^lt_jt_k;p,q\right)\right)\Gamma\left(t_jt_k;p,q,t\right),$$

since

$$\Gamma(pqtz;p,q,t) \;=\; \Gamma(z^{-1};p,q,t).$$

4.3 Second class of dualities

The second class of dual magnetic theories has the same flavour group as in the previous case but different representation content. Again, we expect that there are 35 dual models in this class whose generic representative is described in Table 9.

Table 9.

	$SP(2N)$	$SU(4)$	$SU(4)$	$U(1)_B$	$U(1)$	$U(1)_R$
q	f	$\overline{f}$	1	1	$-\frac{N-1}{4}$	$\frac{1}{4}$
$\widetilde{q}$	f	1	$\overline{f}$	-1	$-\frac{N-1}{4}$	$\frac{1}{4}$
Y	T_A	1	1	0	1	0
M_J	1	f	f	0	$\frac{2J-N+1}{2}$	$\frac{1}{2}$
$\widetilde{V}$	adj	1	1	0	0	$\frac{1}{2}$

Similarly to the previous case, we find

$$\begin{aligned} i_M(p,q,z,\widetilde{y},\widehat{y}) &= -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{SP(2N),adj}(z) \\ &+\frac{1}{(1-p)(1-q)}\Bigg\{(pq)^{r_Y}\chi_{SP(2N),T_A}(z)-(pq)^{1-r_Y}\chi_{SP(2N),\overline{T}_A}(z) \\ &+(pq)^{r_q}v\chi_{SU(4),\overline{f}}(\widetilde{y})\chi_{SP(2N),f}(z)-(pq)^{1-r_q}\frac{1}{v}\chi_{SU(4),f}(\widetilde{y})\chi_{SP(2N),\overline{f}}(z) \\ &+(pq)^{r_{\widetilde{q}}}\frac{1}{v}\chi_{SU(4),\overline{f}}(\widehat{y})\chi_{SP(2N),f}(z)-(pq)^{1-r_{\widetilde{q}}}v\chi_{SU(4),f}(\widehat{y})\chi_{SP(2N),\overline{f}}(z) \\ &+\sum_{J=0}^{N-1}\Big((pq)^{r_{M_J}}\chi_{SU(4),f}(\widetilde{y})\chi_{SU(4),f}(\widehat{y})-(pq)^{1-r_{M_J}}\chi_{SU(4),\overline{f}}(\widetilde{y})\chi_{SU(4),\overline{f}}(\widehat{y})\Big)\Bigg\}, \end{aligned} \tag{30}$$

where

$$r_q=r_{\widetilde{q}}=\frac{1}{4}-\frac{N-1}{4}s,\quad r_Y=s,\quad r_{M_J}=\frac{1}{2}-\frac{1}{2}(N-1-2J)s.$$

Then the index for this magnetic theory is given by

$$I_M^{(2)} = \Gamma((pq)^s;p,q)^{N-1}\prod_{J=0}^{N-1}\prod_{i=1}^{4}\prod_{j=5}^{8}\Gamma((pq)^{r_{M_J}}y_iy_j;p,q)$$
$$\times \frac{(p;p)_\infty^N(q;q)_\infty^N}{2^N N!}\int_{\mathbb{T}^N}\prod_{1\leq i<j\leq N}\frac{\Gamma((pq)^s z_i^{\pm1}z_j^{\pm1};p,q)}{\Gamma(z_i^{\pm1}z_j^{\pm1};p,q)}$$
$$\times \prod_{j=1}^{N}\frac{\prod_{i=1}^4\Gamma((pq)^{r_q}v^2y_i^{-1}z_j^{\pm1};p,q)\prod_{i=5}^8\Gamma((pq)^{r_{\widetilde{q}}}v^{-2}y_i^{-1}z_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{2\pi i z_j}, \quad (31)$$

where we have chosen the same relations between the characteristic variables $v, \widetilde{y}_j, \widehat{y}_j$ and y_j as for $I_M^{(1)}$. In order to prove $I_E = I_M^{(2)}$, it is necessary to repeat the Rains transformation twice with the parameters s_3, s_4, s_5, s_6 playing the role of t_1, t_2, t_3, t_4 in the same way as was done in the $N = 1$ rank case.

4.4 Third dual picture

Finally, there is only one representative in the third class of magnetic theories. It has the symmetry groups

$$G = SP(2N), \qquad F = SU(8)\times U(1),$$

and its fields content is fixed in Table 10.

Table 10.

	$SP(2N)$	$SU(8)$	$U(1)$	$U(1)_R$
q	f	$\overline{f}$	$-\frac{N-1}{4}$	$\frac{1}{4}$
Y	T_A	1	1	0
M_J	1	T_A	$\frac{2J-N+1}{2}$	$\frac{1}{2}$
$\widetilde{V}$	adj	1	0	$\frac{1}{2}$

This dual theory was constructed originally in [14].

The single particle state index in this case is

$$i_M(p,q,z,y) = -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{SP(2N),adj}(z) \quad (32)$$
$$+\frac{1}{(1-p)(1-q)}\Bigg\{(pq)^{r_Y}\chi_{SP(2N),T_A}(z)-(pq)^{1-r_Y}\chi_{SP(2N),\overline{T}_A}(z)$$
$$+(pq)^{r_q}\chi_{SU(8),\overline{f}}(y)\chi_{SP(2N),f}(z)-(pq)^{1-r_q}\chi_{SU(8),f}(y)\chi_{SP(2N),\overline{f}}(z)$$
$$+\sum_{J=0}^{N-1}\Big((pq)^{r_{M_J}}\chi_{SU(8),T_A}(y)-(pq)^{1-r_{M_J}}\chi_{SU(8),\overline{T}_A}(y)\Big)\Bigg\},$$

where

$$r_q = \frac{1-s(N-1)}{4}, \quad r_Y = s, \quad r_{M_J} = sJ+\frac{1-s(N-1)}{2}.$$

The magnetic superconformal index has the form

$$I_M^{(3)} = \Gamma((pq)^{r_Y};p,q)^{N-1} \prod_{J=0}^{N-1} \prod_{1\le i<j\le 8} \Gamma((pq)^{r_{M_J}} y_i y_j;p,q)$$
$$\times \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \int_{\mathbb{T}^N} \prod_{1\le i<j\le N} \frac{\Gamma((pq)^{r_Y} z_i^{\pm 1} z_j^{\pm 1};p,q)}{\Gamma(z_i^{\pm 1} z_j^{\pm 1};p,q)}$$
$$\times \prod_{j=1}^{N} \frac{\prod_{i=1}^8 \Gamma((pq)^{r_q} y_i^{-1} z_j^{\pm 1};p,q)}{\Gamma(z_j^{\pm 2};p,q)} \frac{dz_j}{2\pi i z_j}, \tag{33}$$

Equality $I_E = I_M^{(3)}$ follows from a triple application of the key identity (26) similar to the $N=1$ case considered earlier. For a special quantized value of the parameter $s = \frac{1}{N+1}$, this result describes equality of superconformal indices in the Kutasov-Schwimmer dual models with the $SP(2N)$ gauge group, the number of flavour $N_f = 4$, and a special value of the corresponding parameter $k = N$, see [8, 9, 6]. After a reduction to the s-confining theory (see below), one obtains equality of indices for $N_f = 3$, $k = N$ case as well. As to the 't Hooft anomaly matching conditions for our new dual models – we have verified that all of them are satisfied.

5 Reduction to six flavours

If we take $t_7 t_8 = pq$ (or $y_7 y_8 = (pq)^{1/2}$) for the $SP(2)$-group case, then, because of the reflection identity $\Gamma(a,b;p,q) = 1$ for $ab = pq$, the integral $I(t_1,\ldots,t_8;p,q)$ is reduced to the left-hand side of (1). In physical terms this means that we add to the $SP(2)$ gauge group SQCD Lagrangian mass terms for two components of the quark superfields and tend their masses to infinity washing away them from the spectrum. As to the integral $I_M^{(1)}$, in this limit two pairs of poles pinch the contour of integration $\mathbb{T}$ and integral's value is given by the sum of corresponding residues which yields the right-hand side expression in (1). Physically this means that the corresponding dual magnetic theory is the Wess-Zumino model of appropriate meson fields, and the electric theory has confinement.

However, if we set $y_4 y_5 = (pq)^{1/2}$, then the integral $I_M^{(1)}$ gets simplified, but there is no pinching of the contour and there remains a nontrivial integral. Physically this means that addition of large mass terms to different quark superfield components reduces the number of flavours to 6, but it keeps the gauge group $SP(2)$ intact with the flavour group being reduced to $SU(3)_l \times SU(3)_r \times U(1)_B \times U(1)_{add}$. Note that the latter group is of rank 6 whereas the electric theory has $SU(6)$ flavour group of rank 5.

For the second class dual models the situation is different. For $y_7 y_8 = (pq)^{1/2}$ there is no pinching of the contour in $I_M^{(2)}$. This integral gets simplified, but remains a non-trivial integral. The corresponding SQCD model has the non-trivial gauge group $G = SP(2)$ and $F = SU(4) \times SU(2) \times SU(2)_{add} \times U(1)_B$. Again, this flavour group has rank 6. Vice versa, for $y_4 y_5 = (pq)^{1/2}$ one finds pinching of the contour in $I_M^{(2)}$, the integration disappears, and one comes to the s-confinement with the plain meson fields theory. In

the third class dual model there is only one option – for any $y_j y_k = (pq)^{1/2}$ the contour in $I_M^{(3)}$ is pinched, gauge group disappears, and one comes to the s-confinement.

Similar picture holds for $SP(2N)$, $N > 1$, gauge group case. Skipping the details, we present the corresponding non-trivial field theories in the tables below. The electric theory is described in Table 11.

Table 11.

	$SP(2N)$	$SU(6)$	$U(1)$	$U(1)_R$
Q	f	f	$-\frac{N-1}{3}$	$\frac{1}{6}$
X	T_A	1	1	0
V	adj	1	0	$\frac{1}{2}$

The first class dual models with nontrivial gauge group are described in Table 12. Equality of the corresponding superconformal indices is obtained after mere substitution of the constraint $y_4 y_5 = (pq)^{(1+(N-1)s)/2}$ into formulas (24) and (29).

Table 12.

	$SP(2N)$	$SU(3)$	$SU(3)$	$U(1)$	$U(1)_B$	$U(1)_{add}$	$U(1)_R$
q	f	f	1	$-\frac{N-1}{3}$	-1	-1	$\frac{1}{6}$
$\widetilde{q}$	f	1	f	$-\frac{N-1}{3}$	1	1	$\frac{1}{6}$
M_{1J}	1	$T_A = \overline{f}$	1	$J - 2\frac{N-1}{3}$	4	0	$\frac{1}{3}$
N_{1J}	1	f	1	$J - \frac{N-1}{3}$	2	2	$\frac{2}{3}$
M_{2J}	1	1	$T_A = \overline{f}$	$J - 2\frac{N-1}{3}$	-4	0	$\frac{1}{3}$
N_{2J}	1	1	f	$J - \frac{N-1}{3}$	-2	-2	$\frac{2}{3}$
Y	T_A	1	1	1	0	0	0
$\widetilde{V}$	adj	1	1	0	0	0	$\frac{1}{2}$

The second class dual models with the nontrivial gauge group are described in Table 13. Equality of the corresponding indices is obtained after substitution of the constraint $y_7 y_8 = (pq)^{(1+(N-1)s)/2}$ into formulas (24) and (31).

Table 13.

	$SP(2N)$	$SU(4)$	$SU(2)_{add}$	$SU(2)$	$U(1)$	$U(1)_B$	$U(1)_R$
q	f	$\overline{f}$	1	1	$-\frac{N-1}{3}$	-1	$\frac{1}{6}$
$\tilde{q}$	f	1	f	1	$-\frac{N-1}{3}$	2	$\frac{1}{6}$
M_J	1	f	f	1	$J - \frac{N-1}{3}$	-1	$\frac{2}{3}$
N_J	1	f	1	f	$J - 2\frac{N-1}{3}$	1	$\frac{1}{3}$
Y	T_A	1	1	1	0	0	0
$\widetilde{V}$	adj	1	1	0	0	0	$\frac{1}{2}$

Finally, the field content of the model without gauge group is fixed in Table 14, where $k = 2, \ldots, N$ and $J = 0, \ldots, N-1$.

Table 14.

	$SU(6)$	$U(1)$	$U(1)_R$
M_k	1	k	0
N_J	T_A	$J - 2\frac{N-1}{3}$	$\frac{1}{3}$

For completeness, we present explicitly equality of superconformal indices for this case in the appropriate notation:

$$I_E = \frac{(p;p)_\infty^N (q;q)_\infty^N}{2^N N!} \Gamma(t;p,q)^{N-1} \int_{\mathbb{T}^N} \prod_{1\le j<k\le n} \frac{\Gamma(t z_j^{\pm 1} z_k^{\pm 1};p,q)}{\Gamma(z_j^{\pm 1} z_k^{\pm 1};p,q)}$$

$$\times \prod_{j=1}^N \frac{\prod_{m=1}^6 \Gamma(t_m z_j^{\pm 1};p,q)}{\Gamma(z_j^{\pm 2};p,q)} \frac{dz_j}{2\pi i z_j}$$

$$= I_M = \prod_{j=2}^N \Gamma(t^j;p,q) \prod_{J=0}^{N-1} \prod_{1\le k<m\le 6} \Gamma(t^J t_k t_m;p,q), \qquad (34)$$

where $|p|, |q|, |t|, |t_m| < 1$, and $t^{2n-2}\prod_{m=1}^6 t_m = pq$. This relation describes the elliptic analogue of the Selberg integral for the BC_N-root system [1]. The dual theories of Tables 12 and 13 are new, and the s-confined model of Table 14 was described in [16].

Let us discuss now the possible number of $N_f = 6$ dual models. To count them one has to describe the group structure of integrals remaining after imposing the constraint $t_7 t_8 = pq$. In the notation used for the description of $W(E_7)$ in the end of Sect. 3.1, it is equivalent to the constraint $x_7 + x_8 = const$. This reduces the E_7 root system to E_6. The Weyl group $W(E_6)$ includes the evident $S_6 \times S_2$ group permuting first six and last two coordinates of $x = (x_1, \ldots, x_6; x_7, x_8)$ between themselves. It is generated by the R_v-reflections for the vectors $v \in \pm(e_i - e_j)$ for $1 \le i < j \le 6$ or $i = 7$, $j = 8$. Other nontrivial 20 vectors are obtained by the R_v-reflections of the $S_6 \times S_2$ orbit of x for the vectors

$$v \in \frac{1}{2}\left(\sum_{k=1}^8 (-1)^{\mu_k} e_k\right), \quad \mu_k \in \{0,1\}, \quad \sum_{k=1}^6 \mu_k = 3, \quad \mu_7 + \mu_8 = 1 \qquad (35)$$

leading to the coordinate tranformations

$$x'_j = x_j - \frac{1}{4}\sum_{k=1}^8 (-1)^{\mu_j + \mu_k} x_k,$$

where $j = 1, \ldots, 8$. The remaining 15 nontrivial vectors of the $W(E_6)$-orbit are obtained by the reflections $R_v R_{v'}$ with v, v' from (35). They have coordinates of the form

$$x'_{k_1,k_2,k_3,k_4} = -x_{k_1,k_2,k_3,k_4} + \frac{1}{2}(x_{k_1} + x_{k_2} + x_{k_3} + x_{k_4}),$$

$$x'_{k_5} = -x_7 + \frac{1}{2}(x_7 + x_8 + x_{k_5} + x_{k_6}), \quad x'_{k_6} = -x_8 + \frac{1}{2}(x_7 + x_8 + x_{k_5} + x_{k_6}),$$

where $k_1, \ldots, k_6 \in \{1, \ldots, 6\}$ for $k_i \ne k_j$ and, finally,

$$x'_7 = -x_{k_5} + \frac{1}{2}(x_7 + x_8 + x_{k_5} + x_{k_6}), \quad x'_8 = -x_{k_6} + \frac{1}{2}(x_7 + x_8 + x_{k_5} + x_{k_6}).$$

Since $\dim\{W(E_6)/(S_6 \times S_2)\} = 36$, it is expected that there are 36 dual models with the nontrivial gauge group $G = SP(2N)$ and $N_f = 6$. The rest of 36 dual models with

$N_f = 8$ reduce for $N_f = 6$ to one additional 37th s-confined dual model without gauge group (which becomes completely Higgsed).

There is an interesting problem of comparing anomalies for $N_f = 6$ theories. It is not difficult to check validity of 't Hooft's criterion for the electric and confined theories. However, the first and second class dual models have rather different flavour groups explicitly seen in UV. To compare with the electric theory, one can check first that all anomalies associated with $SU(2)_{add}$ and $U(1)_{add}$ groups vanish. Then it is necessary to embed the remaining parts of the magnetic flavour groups into $SU(6)\times U(1)$ and match the corresponding anomalies in the standard way. The missing anomalies for the cosets $SU(6)/(SU(3) \times SU(3) \times U(1)_B)$ and $SU(6)/(SU(4) \times SU(2) \times U(1)_B)$ may, probably, be imitated by some nonlinear chiral models added to the corresponding SQCD's. If we compare anomalies of the first and second class dual magnetic models between themselves, it is necessary to go further and split both flavour groups without $SU(2)_{add}$ and $U(1)_{add}$ pieces to the smaller subgroup $SU(3) \times SU(2) \times U(1)_1 \times U(1) \times U(1)_B$, for which the anomalies match in the standard way. The rest of the anomalies for non-explicit pieces of the flavour symmetries may, probably, be related to (unknown) non-linear chiral models incorporated into both magnetic theories.

6 Conclusion

To conclude, in this paper we have used known $W(E_7)$-group transformation identities for elliptic hypergeometric integrals in order to describe some known and new Seiberg dualities for $\mathcal{N} = 1$ supersymmetric field theories with $SP(2N)$ gauge groups and the number of flavours $N_f = 8$ and $N_f = 6$. We expect that there are 72 self-dual theories for $N_f = 8$, among which only four have essentially different field content and symmetry groups. For $N_f = 6$ we expect existence of 36 dual theories with the non-trivial gauge group (with only three essentially different field content models) and one s-confined meson fields theory. The flavour groups for $N = 1$ and $N > 1$ differ from each other. The tables for $N = 1$ can be obtained from those of $N > 1$ after setting $J = 0$, $N = 1$ and deleting one row and one column. We decided to give separate consideration of the $N = 1$ case because *all* superconformal indices for dual theories known to us involve generalizations of one or another transformation of the corresponding electric theory characteristic variables. For instance, there is an interesting reduced form of the multiple duality phenomenon for $G = SU(N)$ gauge groups for $N > 2$ [15, 19].

It turns out that the connection of superconformal indices with the elliptic hypergeometric integrals leads to some new results in the theory of elliptic hypergeometric functions. Namely, there are new conjectures for both – the elliptic beta integrals and transformation identities for higher order elliptic hypergeometric functions on root systems. For example, there is an almost complete match of the list of s-confining theories in [16] and elliptic beta integrals on root systems listed in [1], with one of the known integrals leading to a new example of s-confining theories [19]. Vice versa, e.g., an analysis of the s-confining duality for the exceptional gauge group G_2 of [13] leads to the

following new elliptic beta integral

$$\frac{(p;p)_\infty^2(q;q)_\infty^2}{2^2 3}\int_{\mathbb{T}^2}\frac{\prod_{k=1}^3\prod_{m=1}^5\Gamma(t_m z_k^{\pm1};p,q)}{\prod_{1\leq j<k\leq 3}\Gamma(z_j^{\pm1}z_k^{\pm1};p,q)}\prod_{k=1}^2\frac{dz_k}{2\pi i z_k}$$
$$=\prod_{m=1}^5\frac{\Gamma(t_m^2;p,q)}{\Gamma(t_m;p,q)\Gamma((pq)^{1/2}t_m;p,q)}\prod_{1\leq l<m\leq 5}\frac{\Gamma(t_l t_m;p,q)}{\Gamma((pq)^{1/2}t_l t_m;p,q)},\quad (36)$$

where $z_1z_2z_3=1$, $|t_m|<1$, and $\prod_{m=1}^5 t_m=(pq)^{1/2}$. As we have known from a private communication, this formula was conjectured also earlier by M. Ito. At the moment, no proof of this relation is known to the authors.

The considerations of [2, 3, 6] justify the superconformal index building algorithm only for marginally deformed free theories (we are indebted to F. Dolan and Yu. Nakayama for stressing to us this point). However, we apply it to the interacting theories and, by some deep reason, it works for them rather well. Therefore it is necessary to find a more rigorous derivation of formula (3) for Seiberg dual theories.

Consider now the constraints on the parameters of our models coming from the renormalization group analysis. The original Seiberg duality [5] is based on the gauge groups $G_E=SU(N)$ and $G_M=SU(N_f-N)$ with the flavour group $SU(N_f)_l\times SU(N_f)_r\times U(1)_B$ (we used above different counting of the number of flavours which corresponds to $2N_f=6,8$ in the Seiberg notation). Existence of the asymptotic freedom in the electric theory leads to the constraint $N_f<3N$. Similar requirement for the magnetic theory yields the bound $3N/2<N_f$. The combination of two restrictions is called the conformal window. Formally, for $N=2$ and $N_f=4$ the corresponding models lie in the conformal window. However, for all our theories the lower bound $3N/2<N_f$ is not relevant. In the context of $SP(2N)\leftrightarrow SP(2(N_f-N-2))$ duality with $SU(2N_f)$ flavour groups found in [12], the conformal window has the form $3(N+1)/2<N_f<3(N+1)$. Formally, for $N=1$ and $N_f=4$ we have again a pair of models satisfying this constraint, but the lower bound of this window is not relevant again. The reason for the absence of lower bounds stems from the self-duality of our models. Indeed, for all of them the rank of the dual gauge group is fixed, and it does not depend on the number of flavours. As a result, all our $G=SP(2N)$ models are simultaneously automatically asymptotically free (both, for $N=1$ and $N>1$). If we consider these models with arbitrary number of flavours $2N_f$ in the fundamental representation, then for all of them the one loop beta function is $\beta(g)=-g^3(2N+4-N_f)/8\pi^2$. Asymptotic freedom is guaranteed by the universal bound $N_f<2N+4$, which is satisfied in our case $N_f=4$ for arbitrary $N\geq 1$. Let us remark also that at the infrared fixed point, the dimensions of gauge-invariant scalar fields Δ are determined by the R-charges, $\Delta=3R/2$. All our meson fields have thus the dimensions $3/2$ satisfying the unitarity constraints $\Delta\geq 1$.

As to the 't Hooft anomaly matching conditions – they are satisfied pairwise for all dual theories described above for smaller flavour groups. It looks like that the key properties needed for this matching are encoded into the balancing conditions and the $SL_\tau(2;\mathbb{Z})$ or $SL_\sigma(2;\mathbb{Z})$ modular group invariance of "totally" elliptic functions hidden in the structure of superconformal indices [1]. (Here the modular variables τ and σ are related to p and q as $p=e^{2\pi i\tau}$ and $q=e^{2\pi i\sigma}$.)

As a final remark, we would like to speculate on the relevance of the exceptional root system E_7. The well known Kramers-Wannier duality relates 2D Ising models for low and high temperatures. Existence of the unifying model with the Hamiltonian allowing for an explicit transformation of relevant degrees of freedom makes this duality easy to understand. Putting Seiberg duality in a similar context, it looks like that the global symmetry group of the "brane" (higher-dimensional) theory unifying all the Seiberg dual theories for $SP(2N)$ groups is E_7, and it is different "degenerations" that lead to either $SU(8)\times U(1)$ or $SU(4)_l\times SU(4)_r\times U(1)_B\times U(1)$ flavour groups. In any case, the brane dynamics reproducing Seiberg duality for $SU(2)$ gauge group is expected to be more complicated than that described in [25]. In order to clarify the origins of this picture it is necessary to build superpotentials and nonlinear chiral models for our dualities distinguishing them from each other (like in the triality of [10]) and to find their place within the AdS/CFT correspondence framework.

Acknowledgments

V.S. is indebted to A.M. Povolotsky for drawing attention to paper [6] and to V.A. Rubakov for many helpful advises. G.V. would like to thank D.I. Kazakov and A.F. Oskin for valuable discussions. We are grateful to F.A. Dolan and H. Osborn for detailed remarks on the first version of this paper. In particular, we thank H. Osborn for a question on the structure of the coset $W(E_7)/S_8$ which helped us to count the number of new dualities. We thank also A. Kmelnitsky for clarifying flavour group structures for some of the dual models and M. Shifman for the indication on a possible way to resolve the anomaly matching puzzle. The organizers of IV-th Sakharov conference on physics (Moscow, May 2009), Conformal field theory workshop (Chernogolovka, June 2009), and XVI-th International congress on mathematical physics (Prague, August 2009) are thanked for giving to us an opportunity to present the results of our work at these meetings.

The work of V.S. is partially supported by RFBR grant no. 09-01-00271. The work of G.V. is partially supported by the Dynasty foundation, RFBR grant no. 08-02-00856 and grant of the Ministry of Education and Science of the Russian Federation no. 1027.2008.2.

7 Appendix. Characters for unitary and symplectic groups

A character $\chi_R(g)$ for $g\in SU(N)$ is a function of the complex eigenvalues of g

$$x=(x_1,\ldots,x_N),\quad \prod_{i=1}^{N}x_i=1.$$

The characters of the fundamental and antifundamental representations of $SU(N)$ group are given by

$$\chi_{SU(N),f}(x) \ = \ \sum_{i=1}^{N} x_i, \quad \chi_{SU(N),\overline{f}}(x) \ = \ \chi_{SU(N),f}(x^{-1}).$$

We use also general properties of the characters

$$\chi_{f_1 \oplus f_2} = \chi_{f_1} + \chi_{f_2}, \quad \chi_{f_1 \otimes f_2} = \chi_{f_1}\chi_{f_2}.$$

For the adjoint representation one has $\chi_{SU(N),adj}(x) = (\sum_{i=1}^N x_i)(\sum_{j=1}^N x_j^{-1}) - 1$. The character for the anti-symmetric tensor representation of $SU(N)$ is

$$\chi_{SU(N),T_A}(x) \ = \ \sum_{1 \le i < j \le N} x_i x_j, \quad \chi_{SU(N),\overline{T}_A}(x) \ = \ \chi_{SU(N),T_A}(x^{-1}).$$

A character $\chi_R(g)$ for $g \in SP(2N)$ is a function of the complex eigenvalues of g, $x = (x_1, \ldots, x_N)$. The characters of the fundamental and antifundamental representations of $SP(2N)$ group have the form

$$\chi_{SP(2N),f}(x) \ = \chi_{SP(2N),\overline{f}}(x) = \ \sum_{i=1}^{N} (x_i + x_i^{-1}).$$

The character for the adjoint representation of $SP(2N)$ is

$$\chi_{SP(2N),adj}(x) \ = \ \sum_{1 \le i < j \le N} (x_i x_j + x_i x_j^{-1} + x_i^{-1} x_j + x_i^{-1} x_j^{-1}) + \sum_{i=1}^{N} (x_i^2 + x_i^{-2}) + N.$$

For $N = 1$ it coincides with the adjoint representation character for $SU(2)$ group. The character for the anti-symmetric tensor representation of $SP(2N)$ is

$$\chi_{SP(2N),T_A}(x) \ = \ \sum_{1 \le i < j \le N} (x_i x_j + x_i x_j^{-1} + x_i^{-1} x_j + x_i^{-1} x_j^{-1}) + N - 1.$$

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